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1982 J. Phys. A: Math. Gen. 15 L343

(http://iopscience.iop.org/0305-4470/15/7/003)

J. Phys. A: Math. Gen. 15 (1982) L343–L346. Printed in Great Britain

LETTER TO THE EDITOR

Lie-Bäcklund transformation, Painlevé transcendent for a KdV equation with explicit x dependence

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Received 5 March 1982

Abstract. The Lie–Bäcklund transformation is calculated for a KdV equation with explicit space dependence, for which the Lax equation has a moving eigenvalue in the t plane. The corresponding similarity variable and the Painlevé equation are deduced. The role of the Lie–Bäcklund generators in relation to the conservation laws is also discussed.

Over the past few years there has been intense activity regarding the understanding of the complete integrability of nonlinear partial differential equations sustaining solitary wave solutions (Bullough and Caudrey 1980). But the main domain of activity was restricted to equations without explicit (x, t) dependence, except for some singular attempts (Calogero 1980, Roy Chowdhury and Roy 1980). Interesting features of such equations are revealed through the analysis of Lie-Bäcklund type. Here we report such a calculation in relation to a KdV equation with non-uniformity.

Such an equation reads

$$u_t + \gamma u + \alpha x u_x + 6uu_x + u_{xxx} = 0. \tag{1}$$

An initial attempt at solution of such an equation was made by Hirota and Satsuma (1976) by the method of bilinearisation. The eigenvalue problem which could be associated with equation (1) can be written as

$$\begin{gathered}(\partial^2/\partial x^2 + u(x, t))\psi = \lambda\psi, \\ \frac{\partial\psi}{\partial t} + \left[4\frac{\partial^3}{\partial x^3} + 3\left(\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\right) + \alpha x\frac{\partial}{\partial x}\right]\psi = 0,\end{gathered} \tag{2}$$

where the eigenvalue parameter λ is not constant but is governed by

$$\partial\lambda/\partial t + 2\alpha\lambda = 0 \tag{3}$$

for the case $\gamma = 2\alpha$. Let us now consider the change of variable $x \to x^*$, $u \to u^*$, $t \to t^*$ where

$$x^* = x + \varepsilon\xi(x, t, u), \qquad t^* = t + \varepsilon\tau(x, t, u), \qquad u^* = u + \varepsilon\eta(x, t, u). \tag{4}$$

If we demand invariance of our equation under consideration by such a point transformation then

$$u^*_{t^*} + \gamma u^* + \alpha x^* u^*_{x^*} + 6u^* u^*_{x^*} + u^*_{x^*x^*x^*} = 0. \tag{5}$$

Let us denote the total derivative of any function with respect to any variable by superscripts, such as ξ^x, n^t, τ^u etc, and partial derivatives by subscripts. Then we obtain, following the notation of Bluman and Cole (1974),

$$u^*_{x^*} = u_x + \varepsilon\eta^x, \qquad u^*_{x^*x^*} = u_{xx} + \varepsilon\eta^{xx}, \qquad u^*_{x^*x^*x^*} = u_{xxx} + \varepsilon\eta^{xxx}, \tag{6}$$

where the ith-order total derivative of η, $\eta^{xxx\cdots i\text{ factors}}$, is determined from the $(i-1)$th one through the following recurrence relation:

$$\eta^{xxx\cdots i\text{ times}} = \frac{\mathrm{d}}{\mathrm{d}x}\eta^{xxx\cdots(i-1)} - (\xi_x + \xi_u u_x)u_{x^{n+1}} - (\tau_x + \tau_u u_x)u_{x^n t}.$$

Substituting in (5), we obtain

$$\begin{aligned}
&\tau_u = \tau_x = 0, \qquad \xi_u = 0, \qquad \eta_{uu} = 0,\\
&\xi_{xx} = 0 = \eta_{ux}, \qquad \tau_t = 3\xi_x,\\
&(\alpha x + 6u)(\tau_t - \xi_x) + 6\eta - \xi_t + \alpha\xi = 0,\\
&\eta_t - \gamma u(\eta_u - \tau_t) + \gamma\eta + \alpha x\eta_x + 6u\eta_x + \eta_{xxx} = 0.
\end{aligned} \tag{7}$$

It is then easy to observe that a simple set of solutions of equation (7) is given by

$$\tau = f(t), \qquad \xi = \lambda_2(t)x + \lambda_3(t), \qquad \eta = \mu_1(t)u + \mu_2(x, t), \tag{8}$$

where each of the functions λ_i, f, μ is governed by

$$\begin{aligned}
&(D - \alpha)\lambda_3 = a_0\,\mathrm{e}^{-\gamma t} = \psi \qquad \text{(say)},\\
&(\tfrac{1}{3}D^2 - \alpha D)f = b_0\,\mathrm{e}^{-(\alpha+\gamma)t} = \chi \qquad \text{(say)},\\
&\mu_1 = -\tfrac{2}{3}f', \qquad 6\mu_2 = \psi(t) + x\chi(t), \qquad \lambda_2' = \alpha f' + \chi(t).
\end{aligned} \tag{9}$$

Explicitly these are written as

$$\begin{aligned}
&\lambda_3 = a_1\,\mathrm{e}^{\alpha t} - [a_0/(\alpha+\gamma)]\,\mathrm{e}^{-\gamma t},\\
&f = b_1 + b_2\,\mathrm{e}^{3\alpha t} - \frac{b_0\,\mathrm{e}^{-(\alpha+\gamma)t}}{\frac{1}{3}(\alpha+\gamma)^2 + \alpha(\alpha+\gamma)},\\
&\mu_1 = -\tfrac{2}{3}\left(3\alpha b_2\,\mathrm{e}^{3\alpha t} + \frac{b_0}{\alpha + \frac{1}{3}(\alpha+\gamma)}\,\mathrm{e}^{-(\alpha+\gamma)t}\right),\\
&6\mu_2 = a_0\,\mathrm{e}^{-\gamma t} + xb_0\,\mathrm{e}^{-(\alpha+\gamma)t},\\
&\lambda_2 = \alpha f - [b_0/(\alpha+\gamma)]\,\mathrm{e}^{-(\alpha+\gamma)t} + \alpha_1.
\end{aligned} \tag{10}$$

The generator of transformations connected with these transformations is

$$X = \xi\partial/\partial x + \tau\partial/\partial t + \eta\partial/\partial u. \tag{11}$$

Different generators can be constructed by assigning different values to the constants of integration involved.

Next let us turn our attention to the construction of the invariants associated with such transformations. The Lagrange equations pertaining to these are

$$\mathrm{d}x/\xi(x, t, u) = \mathrm{d}t/f(t) = \mathrm{d}u/\eta(x, t, u). \tag{12}$$

These are not immediately integrable but can be seen to be equivalent to the following two linear differential equations:

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{2}{3}\frac{f'}{f}u=\frac{\overline{a_0}}{f}\,\mathrm{e}^{-2\alpha t}+x\frac{\overline{b_0}}{f}\,\mathrm{e}^{-3\alpha t},$$
$$\frac{\mathrm{d}x}{\mathrm{d}t}-\frac{f'}{3f}x=\frac{a_1}{f}\,\mathrm{e}^{\alpha t}-\frac{a_0}{(\alpha+\gamma)f}\,\mathrm{e}^{-\gamma t}. \tag{13}$$

Let us denote

$$\sigma(t)=\int\left(a_1\,\mathrm{e}^{\alpha t}-\frac{a_0}{\alpha+\gamma}\,\mathrm{e}^{-2\alpha t}\right)\frac{\mathrm{d}t}{f^{4/3}},$$
$$\chi(t)=\int\left(\frac{\overline{a_0}}{f^{1/3}}\,\mathrm{e}^{-2\alpha t}+\overline{b_0}\,\mathrm{e}^{-3\alpha t}\sigma(t)\right)\mathrm{d}t. \tag{14}$$

Without integrating in the general case, let us consider the case $b_1=b_0=0$, $a_1=b_2=1$, $a_0=0$, $\gamma=2\alpha$, in which case $f=\mathrm{e}^{3\alpha t}$ and $\sigma=-(3\alpha)^{-1}\,\mathrm{e}^{-3\alpha t}$. Now from the theory of partial differential equations, we know that the general solution is obtained as

$$uf^{2/3}=\phi(x/f^{1/3}-\sigma(t)). \tag{15}$$

In the special situation mentioned above we obtain

$$u(x,t)=\mathrm{e}^{-2\alpha t}\phi[x\,\mathrm{e}^{-\alpha t}+(3\alpha)^{-1}\,\mathrm{e}^{-3\alpha t}] \tag{16}$$

which is the form of the similarity solution. Substituting the S form in the evolution equation, (1), it is easily seen that $\phi(\eta)$, where η is the generalised similarity variable,

$$\eta=x\,\mathrm{e}^{-\alpha t}+(3\alpha)^{-1}\,\mathrm{e}^{-3\alpha t},$$

satisfies

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}=c+\phi-3\phi^2, \tag{17}$$

a nonlinear ordinary differential equation known to belong to the Painlevé class I (Ince 1947). So even when the Lax pair equations have (a) explicit x dependence, (b) time-dependent eigenvalues, the nonlinear equation can be connected to the Painlevé set in a Lie–Bäcklund way. At this point it is worth noting that in the case $\gamma=2\alpha$ equation (1) reads

$$u_t+2\alpha u+\alpha xu_x+6uu_x+u_{xxx}=0 \tag{18}$$

or

$$\frac{\partial}{\partial t}(u\,\mathrm{e}^{\alpha t})+\frac{\partial}{\partial x}[(\alpha xu+3u^2+u_{xx})\,\mathrm{e}^{\alpha t}]=0,$$

that is in the form of a conservation law with

$$\rho=u\,\mathrm{e}^{\alpha t},\qquad j=(\alpha xu+3u^2+u_{xx})\,\mathrm{e}^{\alpha t}. \tag{19}$$

Now, since (1) is invariant under $X=\xi\partial/\partial x+f\partial/\partial t+\eta\partial/\partial u$, we can generate other conservation laws by applying such X's (which are obtainable by assigning special values to the constants in equation (10)) to ρ and j. Let us write out explicitly such forms of X in detail:

$$
\begin{aligned}
X_1 &= e^{3\alpha t}\partial/\partial t + (e^{\alpha t} + \alpha\, e^{3\alpha t})\partial/\partial x - 2\alpha\, e^{3\alpha t} u\partial/\partial u,\\
X_2 &= \partial/\partial t + (x + e^{\alpha t})\partial/\partial x,\\
X_3 &= -(2\alpha)^{-1}\, e^{-2\alpha t}\partial/\partial x + \tfrac{1}{6}\, e^{-2\alpha t}\partial/\partial u,\\
X_4 &= -(6\alpha^2)^{-1}\, e^{-3\alpha t}\partial/\partial t - x(e^{-3\alpha t}/3\alpha)\partial/\partial x + (-\tfrac{1}{3}\, e^{-3\alpha t} u + \tfrac{1}{6} x\, e^{-3\alpha t})\partial/\partial u
\end{aligned}
\tag{20}
$$

and so on.

For each such generator the corresponding (ρ_i, j_i) are given by $\rho_i = x_i\rho$, $j_i = x_i j$. At this point it is worth mentioning that such conservation laws were also deduced by Calogero (1980) by the spectral method.

Lastly, an important point to note is that in relation to the first equation (2), if we assume

$$\psi = \psi(x\, e^{-\alpha t} + e^{-3\alpha t}/3\alpha,\ \lambda_0\, e^{-2\alpha t})$$

then one easily observes

$$[d^2/d\eta^2 + \phi(\eta)]\psi = \lambda_0\psi \tag{21}$$

where λ_0 is a fixed eigenvalue independent of time, and so a usual Schrödinger eigenvalue problem.

In our above computations we have presented an elaborate analysis of a KdV equation with non-uniformity in a Lie–Bäcklund way, which reveals many interesting properties for an equation containing explicit x dependence. Furthermore, the Bäcklund transformation can now be easily deduced through its connection with the Painlevé equation.

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